Estimates on Pull-in Distances in MEMS Models and other Nonlinear Eigenvalue Problems

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Abstract

Motivated by certain mathematical models for Micro-Electro-Mechanical Systems (MEMS), we give upper and lower L^∞ estimates for the minimal solutions of nonlinear eigenvalue problems of the form $-\Delta u = \lambda f(x)F(u)$ on a smooth bounded domain Ω in $\mathbb{R}^N$. We are mainly interested in the *pull-in distance*, that is the $L^\infty-$norm of the extremal solution u^* and how it depends on the geometry of the domain, the dimension of the space, and the so-called *permittivity profile* f. In particular, our results provide mathematical proofs for various observed phenomena, as well as rigorous derivations for several estimates obtained numerically by Pelesko [17], Guo-Pan-Ward [13] and others in the case of the MEMS non-linearity $F(u) = \frac{1}{(1-u)^2}$ and for power-law permittivity profiles $f(x) = |x|^\alpha$.

1 Introduction

We examine problems of the form

$$\begin{cases} -\Delta u = \lambda f(x)F(u) & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (P_{\lambda,f})$$

where Ω is a bounded domain in $\mathbb{R}^N$, $0 < \lambda$, f is a nonnegative nonzero bounded Hölder continuous function, usually dubbed as the *permittivity profile*, and F is a smooth, increasing, convex nonlinearity on its domain $0 \in D_F \subset \mathbb{R}$, such that $F(0) = 1$ and which blows up at the endpoint of its domain. We shall concentrate on the two cases where either F is superlinear and its domain is $D_F := [0, +\infty)$ in which case F is said to be *a regular nonlinearity*, or when $D_F := [0, 1)$ and $\lim_{u \nearrow 1} F(u) = +\infty$ in which case, we say that F is a *singular non-linearity*. Typical regular nonlinearities are $F(u) = e^u$ or $F(u) = (1 + u)^p$ for $p > 1$, while singular nonlinearities include $F(u) = (1 - u)^{-p}$ for $p > 0$.

We say that a solution u of $(P_{\lambda,f})$ is *classical* provided $\|u\|_{L^\infty} < \infty$ (resp., $\|u\|_{L^\infty} < 1$) if F is a regular (resp., singular) nonlinearity. Note that by elliptic regularity theory, this is equivalent to saying that a classical solution is in $C^{2,\alpha}$ for some $\alpha > 0$.

We shall also need to consider H_0^1-*weak solutions* of $(P_{\lambda,f})$ which are those u in $H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \nabla \phi \, dx = \int_\Omega \lambda f F(u) \phi \, dx \quad \text{for all } \phi \in H_0^1(\Omega). \qquad (1)$$

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It is by now well-known that – regardless whether F is a regular or singular nonlinearity – there exists an extremal parameter $\lambda^* \in (0, +\infty)$ depending on Ω, f and N, and which can be defined as

$$\lambda^*(\Omega, f) := \sup\{\lambda > 0 : (P_{\lambda,f}) \text{ has a classical solution}\},$$

such that $(P_{\lambda,f})$ has a *minimal* classical solution u_λ for every $\lambda \in (0, \lambda^*)$, and no weak solution for $\lambda > \lambda^*$. By a *"minimal solution"* u, we mean one such that any other solution v of $(P_{\lambda,f})$ satisfies $v \geq u$ a.e. in Ω. One can then also show that $\lambda \mapsto u_\lambda(x)$ is increasing on $(0, \lambda^*)$ for each $x \in \Omega$. This allows us to define the *extremal solution* by

$$u^*(x) := \lim_{\lambda \nearrow \lambda^*} u_\lambda(x),$$

which then can be shown to be the unique (weak) solution of $(P_{\lambda^*,f})$.

We shall also need the notion of *stability*. Given a weak solution u of $(P_{\lambda,f})$, we say that u is *stable* (resp., *semi-stable*) provided $\mu_1(\lambda, u) > 0$, (resp., $\mu_1(\lambda, u) \geq 0$) where

$$\mu_1(\lambda, u) := \inf\left\{\int_\Omega (|\nabla\psi|^2 - \lambda f(x)F'(u)\psi^2)dx : \ \psi \in H_0^1(\Omega), \int_\Omega \psi^2 = 1\right\}.$$

Under our assumptions on the nonlinearity F, and whether it is regular or singular, one can show that for all $0 < \lambda < \lambda^*$ the minimal solution u_λ is stable, and consequently that u^* is semi-stable. If in addition, u^* is a classical solution of $(P_{\lambda^*,f})$, then necessarily $\mu_1(\lambda^*, u^*) = 0$, since otherwise one could use the Implicit Function Theorem, in a suitable function space, to obtain solutions to $(P_{\lambda,f})$ for $\lambda > \lambda^*$, which would be a contradiction. On the other hand, one has the following useful result, which was proved by Brezis-Vasquez [1] for regular nonlinearities, and by Ghoussoub-Guo [10] in the case of singular nonlinearities with general permittivity profiles.

Proposition 1.1. *A semi-stable $H_0^1(\Omega)-$weak solution of $(P_{\lambda,f})$ that is not a classical solution can only occur at λ^*, in which case it must be equal to the extremal solution u^*.*

The question of the regularity of the extremal solution has attracted a lot of attention in the last decade. For general regular nonlinearities the extremal solution is classical provided one of the following holds:

- Ω is contained in $\mathbb{R}^N$ with $N \leq 3$ (Nedev, see [16]).
- Ω is a ball in $\mathbb{R}^N$ with $N \leq 9$ (Cabre and Capella, see [2]).

The second result is optimal after one considers $F(u) = e^u$ on the unit ball in $\mathbb{R}^{10}$. It is an open question as to whether for $4 \leq N \leq 9$, there is a regular nonlinearity F and a domain $\Omega \subset \mathbb{R}^N$ on which the corresponding extremal solution is unbounded. In the case of the MEMS model, where $F(u) = (1-u)^{-2}$, it is known that the extremal solution is classical provided $N \leq 7$ and that this result is optimal (see [10]). On the other hand, for any dimension $N > 2$, there exists a singular nonlinearity, namely $F(u) = (1-u)^{-p}$ for some $p := p(N) > 0$, such that the corresponding extremal is not classical (see Chapter 3 of [7]).

In this paper, we are mostly interested in the quantitative aspects of the regularity of the extremal solution u^*, which were initially motivated by the equation

$$\begin{cases} -\Delta u = \frac{\lambda f(x)}{(1-u)^2} & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{$M_{\lambda,f}$}$$

In dimension $N = 2$ this equation models a simple *Micro-Electromechanical-Systems* MEMS device, which roughly consists of a dielectric elastic membrane that is attached to the boundary of Ω, and whose upper surface has a thin conducting film. At a distance of 1 above the undeflected membrane sits a grounded plate, i.e., a plate held at zero voltage. When a voltage $V > 0$ is applied to the thin film of the membrane, it deflects towards the ground plate. After various physical limits of the parameters involved, a dimensional argument and a simplification, ones arrives at $(M_{\lambda,f})$ for the steady state of the membrane. Here λ is proportional

to the applied voltage V and the *permittivity profile* $f(x)$ allows for varying dielectric properties of the membrane.

As seen above, one expects the extremal solution u^* in small dimension N to be bounded away from 1, hence to be a classical solution. Since the parameter λ^* corresponds to the critical voltage beyond which there is a snap-through, and since u^* is the optimal deflection of the membrane, it is therefore important for the design of MEMS devices to know how the critical voltage λ^* and the *pull-in-distance* – defined as $\|u^*\|_{L^\infty}$ – depend on the geometry of the membrane and on the permittivity profile. Several analytical and numerical estimates on λ^* have been derived by Pelesko [17], Guo-Pan-Ward [13], Guo-Ghoussoub [10] and others in the case of the MEMS non-linearity $F(u) = \frac{1}{(1-u)^2}$. On the other hand, only numerical estimates have been obtained for the pull-in distance in the case of power-law (resp., exponential) permittivity profiles $f(x) = |x|^\alpha$ (resp., $f(x) = e^{\alpha x}$). In this paper, we shall see that one can give rigorous proofs and estimates for phenomena, which so far have only been observed numerically by various authors. We shall also include corresponding results for general – not necessarily MEMS-type – nonlinearities.

Here is a brief description of the paper. In section 2, we give upper estimates on the pull-in voltage $\lambda^*(\Omega, f)$ in fairly general situations, which will in turn yield lower bounds on $\|u^*\|_{L^\infty}$. What is remarkable here is that the estimates – which are valid for general nonlinearities – turn out to only depend on the permittivity profiles and not on the domain, nor on the dimension. Actually, they also apply to any reasonable uniformly elliptic operator.

In section 3, we give upper estimates on $\|u^*\|_{L^\infty}$ which are computationally friendly. Just as in the proof of the regularity of u^* in low dimensions, we use the energy estimates on the minimal solutions coupled with L^p to L^∞ Sobolev-type constants related to corresponding linear equations. While the result is satisfactory for exponential nonlinearity, it is not so for the MEMS model, which led us to reconsider this nonlinearity in the case of the ball where more precise L^p to Hölder estimates can be used. We stress here that we are not interested in optimal upper estimates but rather estimates which, if given a specific domain Ω and a nonlinearity F, one can easily obtain some numerical parameters by plotting a function of a single variable – possibly – using a Computer Algebra System.

Section 4 was motivated by an intriguing phenomena observed numerically by Guo-Pan-Ward [13], namely that on a two dimensional disc, the pull-in distance does not depend on the power of the permittivity profile $f(x) = |x|^\alpha$. We prove that this is indeed the case by a simple scaling argument which relates the problem $(P_{\lambda,|x|^\alpha})$ on the unit ball of $\mathbb{R}^N$ to $(P_{\lambda,1})$ (which for simplicity we denote by (P_λ)) on a ball in a fractional dimension $N(\alpha)$. (Note that when f is radial and Ω is the unit ball in $\mathbb{R}^N$, all stable solutions of $(P_{\lambda,f})$ are then radial and hence we can examine the problem in fractional dimensions). One can then easily transfer many results established for (P_λ) to $(P_{\lambda,|x|^\alpha})$. This observation, combined for example with the results of Cabre and Cappella [2], leads to new regularity results for the extremal solution associated with $(P_{\lambda,|x|^\alpha})$.

In section 5, we study the asymptotics in λ, and we obtain upper and lower pointwise bounds on the minimal solutions u_λ, in the case where u^* is singular. The upper estimates are valid on arbitrary domains and we restrict ourselves to radial domains for the lower estimates since more explicit bounds can then be found. For that, we show that $\lambda \mapsto u_\lambda$ is actually convex, and we exploit the fact that both u^* and $\frac{d}{d\lambda} u_\lambda|_{\lambda=\lambda^*}$ are explicitly known in the case where Ω is a ball and u^* is singular.

We now list our main notation. For a nonlinearity F, we denote by a_F the upper bound of the domain D_F, which means that $a_F := \infty$ if F is regular, and $a_F := 1$ if F is singular, in such a way that $D_F := [0, a_F)$.

We shall also associate to F the numbers

$$B_F := \sup_{\tau \in (0,a_F)} \frac{\tau}{F(\tau)} \quad \text{and} \quad C_F := \int_0^{a_F} \frac{d\tau}{F(\tau)}. \tag{2}$$

The ball of radius R centred at x_0 in $\mathbb{R}^N$ will be denoted by $B_R(x_0)$. If $x_0 = 0$ then we omit x_0 and if $R = 1$ then we just write B. Given a set Ω in $\mathbb{R}^N$ we let $|\Omega|$ denote its N-dimensional Lebesgue measure, while ω_N denotes the volume of the unit ball B in $\mathbb{R}^N$. The conjugate index of p will be denoted by p' in such a way that $\frac{1}{p} + \frac{1}{p'} = 1$. For a radial function u we write $u(r) = u(|x|)$. The first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ will be denoted by $\lambda_1(\Omega)$ and the corresponding positive eigenfunction will be ϕ_Ω, assuming the normalization $\int_\Omega \phi_\Omega = 1$.

2 Lower estimates for the $L^\infty-$norm of the extremal solution

This section is devoted to the proof of the following result.

Theorem 2.1. *Suppose F is either a regular or singular nonlinearity and that u^* is the extremal solution of $(P_{\lambda,f})$, which we assume to be classical. Then,*

$$\|u^*\|_{L^\infty} \geq (F')^{-1}\left(\max\left\{\frac{1}{B_F}\frac{\inf_\Omega f}{\sup_\Omega f}, \frac{1}{C_F}\frac{\int_\Omega f\phi_\Omega dx}{\sup_\Omega f}\right\}\right), \tag{3}$$

where we define $(F')^{-1}(z) = 0$ for $z < F'(0)$.

Before proceeding with the proof, we give some applications.

Corollary 2.1. *Suppose f is a non-negative bounded Hölder continuous permittivity profile and that the extremal solution u^* of $(P_{\lambda,f})$ on a bounded domain Ω is regular.*

1. *If $F(u) = \frac{1}{(1-u)^p}$, $p > 0$, then*

$$\|u^*\|_{L^\infty} \geq 1 - \min\left\{\frac{p}{p+1}\left(\frac{\sup_\Omega f}{\inf_\Omega f}\right)^{\frac{1}{p+1}}, \left(\frac{p}{p+1}\frac{\sup_\Omega f}{\int_\Omega f\phi_\Omega\, dx}\right)^{\frac{1}{p+1}}\right\}. \tag{4}$$

   In particular, when the permittivity $f \equiv 1$, then for any dimension $1 \leq N \leq 2 + \frac{4p}{p+1} + 4\sqrt{\frac{p}{p+1}}$, and any bounded domain $\Omega \subset \mathbb{R}^N$, we have

$$\|u^*\|_{L^\infty} \geq \frac{1}{p+1}. \tag{5}$$

2. *If $F(u) = (u+1)^p$, $p > 1$, then*

$$\|u^*\|_{L^\infty} \geq \max\left\{\frac{p}{p-1}\left(\frac{\inf_\Omega f}{\sup_\Omega f}\right)^{\frac{1}{p-1}}, \left(\frac{p-1}{p}\frac{\int_\Omega f\phi_\Omega dx}{\sup_\Omega f}\right)^{\frac{1}{p-1}}\right\} - 1 \tag{6}$$

   In particular, when $f \equiv 1$, then for any dimension $1 \leq N \leq 10$, and any bounded domain $\Omega \subset \mathbb{R}^N$, we have $\|u^*\|_{L^\infty} \geq \frac{1}{p-1}$.

3. *If $F(u) = e^u$, then*

$$\|u^*\|_{L^\infty} \geq \max\left\{1 + \log\left(\frac{\inf_\Omega f}{\sup_\Omega f}\right), \log\left(\frac{\int_\Omega f\phi_\Omega\, dx}{\sup_\Omega f}\right)\right\}. \tag{7}$$

   In particular, when the permittivity $f \equiv 1$, then for any dimension $1 \leq N \leq 9$, and any bounded domain $\Omega \subset \mathbb{R}^N$, we have $\|u^*\|_{L^\infty} \geq 1$.

Remark 2.1. Note that the lower bounds (when $f = 1$) are independent of the domain. It is also fairly easy to adapt the proof below to show that they are not particularly exclusive to the Laplacian $-\Delta$. Indeed, the same lower bounds can be obtained if we replace it by any operator of the form $L(u) := -\mathrm{div}(A(x)\nabla u)$ where $A(x)$ is a symmetric uniformly positive definite $N \times N$ matrix defined in Ω.

Moreover, the same arguments show that the extremal solution associated with

$$\Delta^2 u = \lambda F(u) \qquad \text{on } \Omega, \tag{8}$$

also satisfies the same lower bound, where for general domains Ω we restrict our attention to the Navier boundary conditions: $u = \Delta u = 0$ on $\partial\Omega$, while in the case of Ω being a ball we can use the Dirichlet boundary conditions: $u = \partial_\nu u = 0$ on ∂B. For recent advances on fourth order nonlinear eigenvalue problems, we refer to [3], [4], and [6].

The proof of Theorem 2.1 follows immediately from the combination of the following two propositions. The first provides upper estimates on λ^*, in terms of F, Ω and f.

Proposition 2.1. *Suppose F is either a regular or singular nonlinearity. Then*

$$\lambda^*(\Omega, f) \le \lambda_1(\Omega) \min\left\{ \frac{B_F}{\inf_\Omega f}, \frac{C_F}{\int_\Omega f\phi_\Omega dx} \right\}, \tag{9}$$

where B_F and C_F are given in (2).

Proof. Supposing u is a classical solution of $(P_{\lambda,f})$, we multiply both sides of the equation by ϕ_Ω and integrate to obtain

$$\int_\Omega (\lambda F(u)f - \lambda_1(\Omega)u)\,\phi_\Omega dx = 0.$$

Since $\phi_\Omega > 0$ we must have

$$\lambda \le \lambda_1(\Omega) \sup_\Omega \frac{u}{fF(u)} \le \frac{\lambda_1(\Omega)}{\inf_\Omega f} \sup_{z\in D_F} \frac{z}{F(z)} = \frac{\lambda_1(\Omega)B_F}{\inf_\Omega f}.$$

For the second bound, multiply $(P_{\lambda,f})$ by $\frac{\phi_\Omega}{F(u)}$ and integrate to obtain

$$\begin{aligned}
\int_\Omega \lambda f\phi_\Omega dx &= \int_\Omega (-\Delta u)\frac{\phi_\Omega}{F(u)}dx \\
&= \int_\Omega \frac{\nabla u\cdot\nabla\phi_\Omega}{F(u)}dx - \int_\Omega \frac{\phi_\Omega F'(u)|\nabla u|^2}{F(u)^2}dx \\
&\le \int_\Omega \frac{\nabla u\cdot\nabla\phi_\Omega}{F(u)}dx \\
&= \int_\Omega \nabla\phi_\Omega\cdot\nabla\left(\int_0^{u(x)}\frac{1}{F(\tau)}d\tau\right)dx, \\
&= \lambda_1(\Omega)\int_\Omega \phi_\Omega\left(\int_0^{u(x)}\frac{1}{F(\tau)}d\tau\right)dx \\
&\le \lambda_1(\Omega)C_F,
\end{aligned}$$

after recalling the normalization of ϕ_Ω. ∎

Proposition 2.2. *Suppose u^* is the extremal solution of $(P_{\lambda,f})$ which we assume to be classical. Then*

$$\lambda_1(\Omega) \le \lambda^*\|fF'(u^*)\|_{L^\infty}. \tag{10}$$

Proof. If u is a classical solution of $(P_{\lambda,f})$ with $\lambda_1(\Omega) \ge \lambda\|fF'(u)\|_{L^\infty}$, then the variational formulation of the first eigenvalue $\lambda_1(\Omega)$ of the Laplacian yields

$$\int_\Omega |\nabla\phi|^2dx \ge \lambda_1(\Omega)\int_\Omega \phi^2dx \ge \lambda\|fF'(u)\|_{L^\infty}\int_\Omega \phi^2dx \ge \lambda\int_\Omega fF'(u)\phi^2dx,$$

which means that u is then a stable solution of $(P_{\lambda,f})$.

Assuming now that u^* is a classical solution then, as mentioned in the introduction, we necessarily have that $\mu_1(u^*) = 0$. Using the bifurcation theorem of Crandall-Rabinowitz [5], one can then obtain a second branch of solutions U_λ to $(P_{\lambda,f})$ for λ in a small interval $(\lambda^*-\varepsilon, \lambda^*)$. Moreover these solutions are unstable in the sense that $\mu_1(\lambda, U_\lambda) < 0$. It then follows from the above that $\lambda_1(\Omega) \le \lambda\|fF'(U_\lambda)\|_{L^\infty}$. Sending $\lambda \nearrow \lambda^*$ gives the desired result. ∎

3 Upper estimates for the $L^\infty-$norm of the extremal solution

In this section we look for upper estimates on the extremal solution u^* associated with (P_λ), where F is one of the three linearities considered in Corollary 2.1, and where we take $f(x) = 1$ for simplicity. The methods consist of combining the energy estimates – which are critical in showing that the extremal solution is regular in low dimension – with various L^∞ and Hölder estimates for linear equations.

The following simple observation can be useful when looking for upper estimates.

Observation 3.1. *Suppose u^* is the extremal solution associated with (P_λ) in Ω with extremal parameter λ^*. Then the extremal solution associated with (P_λ) in the domain $\Omega_\rho := \rho\Omega$ (where $\rho > 0$) is given by $v^*_\rho(x) := u^*(\frac{x}{\rho})$ with extremal parameter $\lambda^*(\Omega_\rho) = \frac{\lambda^*(\Omega)}{\rho^2}$.*

3.1 Upper estimates on general domains

We begin with the case of exponential nonlinearities.

Theorem 3.1. *Suppose $F(u) = e^u$, Ω is a bounded domain in $\mathbb{R}^N$ and u^* is the extremal solution associated with (P_λ).*

1. *If $3 \le N \le 9$, then*
$$\|u^*\|_{L^\infty} \le \frac{\lambda_1(\Omega)\beta_N}{e(N-2)}\Big(\frac{|\Omega|}{\omega_N}\Big)^{\frac{N}{2}}, \tag{11}$$
where
$$\beta_N := \inf\Big\{N^{\frac{-1}{2t+1}}\Big(\frac{2t}{4t+2-N}\Big)^{\frac{2t}{2t+1}}\Big(\frac{4}{2-t}\Big)^{\frac{1}{t}};\quad \frac{N-2}{4} < t < 2\Big\}.$$

2. *If $\Omega \subset B_{\frac{1}{2}} \subset \mathbb{R}^2$, then*
$$\|u^*\|_{L^\infty} \le \frac{\lambda_1(\Omega)}{e}\inf\Big\{\Big(\frac{4}{2-t}\Big)^{\frac{1}{t}}\big(\frac{|\Omega|}{2\pi}\big)^{\frac{1}{2t+1}}\Lambda\Big(\frac{2t+1}{2t},\big(\frac{|\Omega|}{\pi}\big)^{\frac{1}{2}}\Big)^{\frac{2t}{2t+1}};\quad 0 < t < 2\Big\}, \tag{12}$$
where we define for $p > 1$ and $0 < R < 1$,
$$\Lambda(p, R) := \int_0^R (-\log(r))^p r dr.$$

Using a computer algebra system one can evaluate the constants and obtain:

$$\beta_3 = 1.9915,\quad \beta_4 = 2.2324,\quad \beta_5 = 2.6689,\quad \beta_6 = 3.42269,$$
$$\beta_7 = 4.81191,\quad \beta_8 = 7.9408166,\quad \beta_9 = 19.0031.$$

Note that one can combine this upper estimate with the previous lower estimate on u^* to obtain the following lower bound on the first eigenvalue of the Laplacian on a bounded domain in $\mathbb{R}^N$ whenever $3 \le N \le 9$:

$$\lambda_1(\Omega) \ge \frac{e(N-2)}{\beta_N}\Big(\frac{\omega_N}{|\Omega|}\Big)^{\frac{N}{2}}. \tag{13}$$

We also consider the case of a MEMS nonlinearity.

Theorem 3.2. *Suppose $F(u) = (1-u)^{-2}$, Ω is a bounded domain in $\mathbb{R}^N$ and u^* is the extremal solution associated with (M_λ) in Ω. If $3 \le N \le 7$, then*

$$\|u^*\|_{L^\infty} \le 1 - e^{-\frac{\lambda_1(\Omega)\gamma_N}{2(N-2)}\big(\frac{|\Omega|}{\omega_N}\big)^{\frac{2}{N}}}, \tag{14}$$

where

$$\gamma_N := \inf\left\{ \frac{16N^{\frac{-3}{2t+3}}}{27}\left(\frac{2t}{4t+6-3N}\right)^{\frac{2t}{2t+3}}\left(\frac{4(2t+1)}{4t+2-t^2}\right)^{\frac{2}{t}};\quad \frac{3(N-2)}{4} < t < 2+\sqrt{6}\right\}.$$

Remark 3.1. *Using a similar approach one can show that if u^* is the extremal solution associated with (P_λ), in the case where $F(u) = (u+1)^p$, $p > 1$, and $N = 3$ or $N = 4$ then*

$$\|u^*\|_{L^\infty} \leq \frac{(p-1)^{p-1}\lambda_1(\Omega)\beta_{N,p}}{p^p(N-2)}\Big(\frac{|\Omega|}{\omega_N}\Big)^{\frac{2}{N}},$$

where

$$\beta_{N,p} = \inf\left\{ \frac{(2tp-p-t^2)^{\frac{-p}{t}}(2t-1)^{\frac{2t-1}{2t+p-1}+\frac{p}{t}}(2p)^{\frac{p}{t}}}{N^{\frac{p}{2t+p-1}}(4t+2p-2-Np)^{\frac{2t-1}{2t+p-1}}} : \max\{t_p^-, t_{N,p}\} < t < t_p^+\right\},$$

and where

$$t_p^- := p-\sqrt{p^2-p},\quad t_p^+ := p+\sqrt{p^2-p},\quad t_{N,p} := \frac{pN}{4}-\frac{p}{2}+\frac{1}{2}.$$

We have omitted $N = 2$ just for simplicity. To obtain estimates for $N \leq 10$ one has to perform a bootstrap argument or restrict the range of values for p.

For proving the above theorems we shall need the following easy lemmas.

Lemma 3.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$.*

1. *If $N \geq 3$ and $\tau > \frac{N}{2}$, then for all $x \in \Omega$,*

$$\left(\int_\Omega \frac{1}{|y-x|^{(N-2)\tau'}}dy\right)^{\frac{1}{\tau'}} \leq \frac{\omega_N^{1-\frac{2}{N}} N^{1-\frac{1}{\tau}}(\tau-1)^{\frac{\tau-1}{\tau}}|\Omega|^{\frac{2}{N}-\frac{1}{\tau}}}{(2\tau-N)^{\frac{\tau-1}{\tau}}}.$$

2. *If $N = 2$ and $\Omega \subset B_{\frac{1}{2}} \subset \mathbb{R}^2$, then for all $x \in \Omega$,*

$$\left(\int_\Omega (-\log(|y-x|))^{\tau'}dy\right)^{\frac{1}{\tau'}} \leq (2\pi)^{\frac{\tau-1}{\tau}}\Lambda\Big(\frac{\tau}{\tau-1}, \frac{|\Omega|^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}\Big)^{\frac{\tau-1}{\tau}}.$$

We now obtain L^∞ bounds on linear equations.

Lemma 3.2. *Suppose $-\Delta u = g(x) \geq 0$ in Ω with $u = 0$ on $\partial\Omega$ where Ω a bounded domain in $\mathbb{R}^N$ and g is smooth.*

1. *If $N \geq 3$, then for all $\tau > \frac{N}{2}$,*

$$\|u\|_{L^\infty} \leq \frac{\|g\|_{L^\tau}(\tau-1)^{\frac{\tau-1}{\tau}}|\Omega|^{\frac{2}{N}-\frac{1}{\tau}}}{N^{\frac{1}{\tau}}(N-2)\omega_N^{\frac{2}{N}}(2\tau-N)^{\frac{\tau-1}{\tau}}}.$$

2. *If $N = 2$ and $\Omega \subset B_{\frac{1}{2}}$, then for all $\tau > 1$,*

$$\|u\|_{L^\infty} \leq \frac{\|g\|_{L^\tau}\Lambda\Big(\frac{\tau}{\tau-1}, \frac{|\Omega|^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}\Big)^{\frac{\tau-1}{\tau}}}{(2\pi)^{\frac{1}{\tau}}}.$$

Proof. In both cases, we let $v(x)$ denote the Newtonian potential of g, i.e.,

$$v(x) := \frac{1}{N(N-2)\omega_N} \int_\Omega \frac{g(y)}{|y-x|^{N-2}} dy,$$

for $N \geq 3$ and

$$v(x) := \frac{1}{2\pi} \int_\Omega (-\log(|y-x|)) g(y) dy,$$

for $N = 2$. Since $0 \leq u(x) \leq v(x)$ in Ω, it suffices to show the desired L^∞ estimate on v. To do this, one uses (for $N \geq 3$) Hölder's inequality to write

$$v(x) \quad \leq \quad \frac{1}{N(N-2)\omega_N} \|g\|_{L^\tau} \left(\int_\Omega \frac{1}{|y-x|^{(N-2)\tau'}} dy \right)^{\frac{1}{\tau'}}.$$

and then use the integral estimate in the previous lemma. $\square$

We now derive the energy estimates for stable solutions.

Lemma 3.3. *Suppose u is a classical semi-stable solution of (P_λ).*

1. *If $F(u) = e^u$, then for all $0 < t < 2$, we have*

$$\|e^u\|_{L^{2t+1}} \leq \Big(\frac{4}{2-t}\Big)^{\frac{1}{t}} |\Omega|^{\frac{1}{2t+1}}.$$

2. *If $F(u) = (1-u)^{-2}$, then for all $0 < t < 2 + \sqrt{6}$, we have*

$$\|(1-u)^{-2}\|_{L^{t+\frac{3}{2}}} \leq \left(\frac{4(2t+1)}{4t+2-t^2} \right)^{\frac{2}{t}} |\Omega|^{\frac{2}{2t+3}}.$$

Proof. 1) Using the test function $\psi := e^{tu} - 1$, where $0 < t < 2$, in the stability conditions gives

$$\frac{\lambda}{t^2} \int_\Omega e^u (e^{tu} - 1)^2 \leq \int_\Omega e^{2tu} |\nabla u|^2.$$

Now testing (P_λ) on $\phi = e^{2tu} - 1$ and rearranging, gives

$$\int_\Omega e^{2tu} |\nabla u|^2 = \frac{\lambda}{2t} \int_\Omega e^u (e^{2tu} - 1).$$

Comparing the last two inequalities and dropping some positive terms gives

$$\left(\frac{1}{t} - \frac{1}{2} \right) \int_\Omega e^{(2t+1)u} \leq \frac{2}{t} \int_\Omega e^{(t+1)u},$$

and after an application of Hölder's inequality on the right one obtains

$$\|e^u\|_{L^{2t+1}} \leq \frac{4^{\frac{1}{t}}}{(2-t)^{\frac{1}{t}}} |\Omega|^{\frac{1}{2t+1}}. \tag{15}$$

2) Take $\psi := (1-u)^{-2} - 1$, $\phi := (1-u)^{-2t-1} - 1$ and proceed as in 1) by putting ψ into the stability condition and testing (M_λ) on ϕ. We obtain

$$\left(\frac{2}{t^2} - \frac{1}{2t+1} \right) \int_\Omega \frac{1}{(1-u)^{2t+3}} \leq \frac{4}{t^2} \int_\Omega \frac{1}{(1-u)^{t+3}},$$

after dropping a couple of positive terms. Hölder's inequality then yields

$$\left(\frac{2}{t^2} - \frac{1}{2t+1} \right) \left\| \frac{1}{1-u} \right\|_{L^{2t+3}}^t \leq \frac{4}{t^2} |\Omega|^{\frac{t}{2t+3}}. \tag{16}$$

$\square$

We now combine the energy estimates with the linear estimates to obtain upper estimates on u^*.

Proof of Theorem 3.1: Use Lemma 3.2 with $g(x) := \lambda^* e^{u^*}$ and $\tau = 2t+1$ along with the estimate $\lambda^* \le \frac{\lambda_1(\Omega)}{e}$ to arrive at an estimate of the form

$$\|u^*\|_{L^\infty} \le C(t,N,|\Omega|)\frac{\lambda_1(\Omega)}{e}\|e^{u^*}\|_{L^{2t+1}},$$

where $C(t,N,|\Omega|)$ is provided by Lemma 3.2. Now replace the L^p-norm on the right using the energy estimates from Lemma 3.3 to arrive at the desired result. The restrictions on t are a result of the restrictions on τ in the linear estimates along with the restrictions on t from the energy estimates. $\Box$

Proof of Theorem 3.2: Let Ω denote a bounded domain in $\mathbb{R}^N$ where $3 \le N \le 7$ and let u^* denote the extremal solution associated with (M_λ) in Ω. Since the reasoning works for any log-convex nonlinearity F (i.e., $u \mapsto \log(F(u))$ is convex), we define $v := \log(F(u^*))$, and so

$$-\Delta v = -\frac{d^2}{du^2}\log(F(u))\Big|_{u=u^*}|\nabla u^*|^2 + \lambda^* F'(u^*) \qquad \text{in } \Omega,$$

with $v=0$ on $\partial\Omega$. Since F is log convex, the first term on the right is negative. We now define w by

$$\begin{aligned} -\Delta w &= \lambda^* F'(u^*) \qquad &&\text{in } \Omega,\\ w &= 0 &&\text{on } \partial\Omega, \end{aligned}$$

and so $0 \le v(x) \le w(x)$ a.e. in Ω by the maximum principle. Using the linear estimates from Lemma 3.2 with $g(x) := \lambda^* F'(u^*)$ one has

$$\|\log\frac{1}{(1-u)^2}\|_{L^\infty} = \|v\|_{L^\infty} \le \|w\|_{L^\infty} \le \tilde{C}_\tau \lambda^* \|F'(u^*)\|_{L^\tau} = \tilde{C}_\tau\lambda^*\left\|\frac{1}{1-u^*}\right\|^3_{L^{3\tau}}.$$

Taking now $\tau = \frac{2t}{3}+1 > \frac{N}{2}$, we can then replace the $L^\tau-$norm on the right by using the energy estimates from Lemma 3.3, which will give the desired conclusion. $\Box$

3.2 Upper estimates on radial domains

While the upper estimate on general domains obtained in the last subsection is quite satisfactory for the exponential nonlinearity, it is not so for the case of the MEMS nonlinearity. Indeed, using Maple one sees that if $\Omega := (0,1)^3$ the unit cube in $\mathbb{R}^3$ (and so $\lambda_1(\Omega) = 3\pi^2$), Formula (14) would then give that

$$\|u^*\|_{L^\infty} \le .993..., \tag{17}$$

which is clearly not a very good upper estimate. This is mainly due to the fact that we drop a potentially large term in the proof of Theorem 3.2, when we replaced v by w in order to apply the linear estimate of Lemma 3.2. Note that this was not needed for the exponential nonlinearity in the proof of Theorem 3.1.

In this section we examine radial domains, where better results are available on u^*, at least in the case of $F(u) = (1-u)^{-2}$. One can also examine the exponential nonlinearity using this approach but we won't do this since the last section seems to give satisfactory results. For simplicity, we shall also restrict our attention to the case of $f \equiv 1$. The main difference is that we use here Hölder estimates on linear equations versus the L^∞ estimates of the last subsection.

For the remainder of this section we assume that Ω is the unit ball B in $\mathbb{R}^N$ and $F(u) = (1-u)^{-2}$. We define the following parameter:

$$\gamma(\tau,N) := \begin{cases} \frac{\tau}{2\tau-1} & N=1\\[2ex] \frac{\tau}{4(\tau-1)} & N=2\\[2ex] \frac{(\tau-1)^{\frac{\tau-1}{\tau}}}{(N-2)N^{\frac{1}{\tau}}(2\tau-N)^{\frac{\tau-1}{\tau}}} & N\ge 3. \end{cases}$$

Lemma 3.4. *Let u denote a smooth radially decreasing solution of $-\Delta u = g(r) \geq 0$ in the unit ball B of $\mathbb{R}^N$. If $\max\{1, \frac{N}{2}\} < \tau < \infty$, then one has the estimate:*

$$u(0) \geq u(R) \geq u(0) - \frac{\gamma(\tau,N)\|g\|_{L^\tau}}{\omega_N^{\frac{1}{\tau}}} R^{2-\frac{N}{\tau}} \quad \textit{for all } R \in (0,1). \tag{18}$$

Proof. When $N = 1$, we integrate the equation between 0 and r, and apply Hölder's inequality to obtain $-u'(r) \leq \frac{\|g\|_\tau r^{\frac{1}{\tau'}}}{2}$. Now integrate both terms between 0 and R, and use again Hölder's inequality to obtain the desired result.

When $N \geq 2$, we multiply the equation by r and integrate over $(0, R)$ to arrive at

$$R(-u'(R)) + (N-2)(u(0) - u(R)) = \int_0^R rg(r)dr.$$

If now $N = 2$, then one has

$$R(-u'(R)) = \int_0^R rg(r)dr \leq \frac{\|g\|_{L^\tau} R^{\frac{2}{\tau'}}}{2\pi^{1-\frac{1}{\tau'}}}.$$

Dividing by R and integrating the result over $(0, R)$ gives the claim.

Now take $N \geq 3$. Since $-u'(R) \geq 0$ we can drop a term to arrive at

$$(N-2)(u(0) - u(R)) \leq \int_0^R rg(r)dr = \frac{1}{N\omega_N} \int_{B_R} \frac{g(x)}{|x|^{N-2}} dx \leq \frac{\|g\|_{L^\tau}}{N\omega_N} \left(\int_{B_R} \frac{1}{|x|^{(N-2)\tau'}} dx \right)^{\frac{1}{\tau'}},$$

and then use Lemma 3.1 to evaluate the integral on the right and finish the proof. □

We now come to the result which will yield our upper estimates on u^*.

Theorem 3.3. *Suppose u is a smooth semi-stable solution of (P_λ) on the unit ball B in $\mathbb{R}^N$, where $1 \leq N \leq 11$. Then, for $\max\left\{0, \frac{N-3}{2}\right\} < t < 2 + \sqrt{6}$, we have*

$$\int_0^1 \frac{R^{N-1}\, dR}{\left(1 - \|u\|_{L^\infty} + \frac{4\lambda_1(B)\gamma(t+\frac{3}{2},N)}{27} \left(\frac{4(2t+1)}{4t+2-t^2}\right)^{\frac{2}{t}} R^{\frac{4t+6-2N}{2t+3}}\right)^{2t+3}} \leq \frac{1}{N} \left(\frac{4(2t+1)}{4t+2-t^2}\right)^{\frac{2t+3}{t}}. \tag{19}$$

Remark 3.2. Note that the above theorem only shows that $\|u\|_{L^\infty}$ is bounded away from 1 if $4t+6-2N \geq 2N-1$ which, once coupled with the other condition on t cannot be satisfied in the higher dimensions. This is to be expected since the extremal solution u^* satisfies $u^*(0) = 1$ for $N \geq 8$.

Proof. Suppose u is a smooth semi-stable (so radial) solution of (P_λ). Then, the above linear estimate applied with $g(r) := \lambda(1-u)^{-2}$, gives that for all $R \in (0,1)$,

$$1 - u(R) \leq 1 - u(0) + \frac{\lambda\gamma(\tau, N)\|(1-u)^{-2}\|_{L^\tau} R^{2-\frac{N}{2}}}{\omega_N^{\frac{1}{\tau}}}.$$

Now use the upper bound $\lambda^* \leq \frac{4\lambda_1(\Omega)}{27}$ from Proposition 2.1, take $\tau = t + \frac{3}{2}$, and replace the L^τ−norm on the right via the energy estimate from Lemma 3.3, to obtain

$$1 - u(R) \leq 1 - u(0) + \frac{4\lambda_1(B)\gamma(t+\frac{3}{2}, N)}{27} \left(\frac{4(2t+1)}{4t+2-t^2}\right)^{\frac{2}{t}} R^{\frac{4t+6-2N}{2t+3}}.$$

This yields the inequality

$$N\omega_N \int_0^1 \frac{R^{N-1}\,dR}{\left(1-\|u\|_{L^\infty}+\frac{4\lambda_1(B)\gamma(t+\frac{3}{2},N)}{27}\left(\frac{4(2t+1)}{4t+2-t^2}\right)^{\frac{2}{t}}R^{\frac{4t+6-2N}{2t+3}}\right)^{2t+3}} \le N\omega_N \int_0^1 \frac{R^{N-1}\,dR}{(1-u(R))^{2t+3}}.$$

But the right hand side is actually equal to $\|(1-u)^{-2}\|_{L^{t+\frac{3}{2}}}^{t+\frac{3}{2}}$, hence we can again use the energy estimate from Lemma 3.3 to majorize it and complete the proof.

□

Remark 3.3. *Using Maple to approximate the integral in (19) while optimizing over t, we get the following estimates on the extremal solution u^* of (P_λ) on the unit ball in $\mathbb{R}^N$.*

1. *If $N=1$, then $\|u^*\|_{L^\infty} \le .49...$*
2. *If $N=2$, then $\|u^*\|_{L^\infty} \le .55...$*

We now obtain some explicit upper bounds on u^* in dimensions $N=1,2$. For that, we define

$$C(t,N) := \frac{4\lambda_1(B)\gamma(t+\frac{3}{2},N)}{27}\left(\frac{4(2t+1)}{4t+2-t^2}\right)^{\frac{2}{t}}.$$

Corollary 3.1. *Suppose u^* is the extremal solution of (P_λ) on the unit ball in $\mathbb{R}^N$.*

1. *If $N=1$, then*

$$\|u^*\|_{L^\infty} \le 1-\sup\left\{\left(2C(t,1)(t+1)\left(\frac{4(2t+1)}{4t+2-t^2}\right)^{\frac{2t+3}{t}}+\frac{1}{C(t,1)^{2+2t}}\right)^{\frac{-1}{2t+2}} : 0<t<2+\sqrt{6}\right\}.$$

2. *If $N=2$, then*

$$\|u^*\|_{L^\infty} \le 1-\sup\left\{\left(C(t,2)^2(t+1)\left(\frac{4(2t+1)}{4t+2-t^2}\right)^{\frac{2t+3}{t}}+\frac{2t+2}{C(t,2)^{2t+1}}\right)^{\frac{-1}{2t+1}} : \frac{1}{2}\le t<2+\sqrt{6}\right\}.$$

Proof. 1) For $0<t<2+\sqrt{6}$ one has $\frac{4t+6-2N}{2t+3}\ge 1$ and so we can replace the power on R in (19) by 1, so as to be able to explicitly calculate the integral in (19). One then drops a few positive terms to arrive at the desired result.

2) For $\frac{1}{2}\le t<2+\sqrt{6}$ one has $\frac{4t+6-2N}{2t+3}\ge 1$, so again we replace the power on R in (19) by 1 and carry on as in the first part. □

4 Effect of power-law profiles on pull-in distances

Our goal in this section is to study the effect of power-like permittivity profiles $f(x)=|x|^\alpha$ on the problem $(P_{\lambda,\alpha})$ (our notation for $(P_{\lambda,|x|^\alpha})$) on the unit ball $B=B_1(0)$. Numerical results – in particular those obtained by Guo, Pan and Ward in [13] for MEMS nonlinearities– give lots of information, but the most intriguing one is their observation that on a 2-dimensional disc, the pull-in distance does not depend on α, at least in the case where $F(u)=(1-u)^{-2}$, and that the solution develops a boundary-layer structure near the boundary of the domain as α is increased. In other words, the $L^\infty-$norm of the extremal solution of $(M_{\lambda,\alpha})$ is independent of $\alpha\ge 0$. In this section, we shall give a simple proof of this observation and other interesting phenomena, which actually holds true for more general nonlinearities.

We first observe that since $r\to r^\alpha$ is increasing, the moving plane method of Gidas, Ni and Nirenberg [12] does not guarantee the radial symmetry of all solutions to $(S_{\lambda,f})$. However, one can show as in [10] the following proposition.

Proposition 4.1. *Let Ω be a radially symmetric domain and assume f is a radial profile on Ω. Then, the minimal solutions of $(P_{\lambda,f})$ on Ω are necessarily radially symmetric and consequently*

$$\lambda^*(\Omega, f) = \lambda^*_r(\Omega, f) = \sup\big\{\lambda;\ (P_{\lambda,f}) \text{ has a radial solution}\big\}.$$

Moreover, if Ω is a ball, then any radial solution of $(P_{\lambda,f})$ attains its maximum at 0.

Proof. It is clear that $\lambda^*_r(\Omega, f) \le \lambda^*(\Omega, f)$, and the reverse will be proved if we establish that every minimal solution of $(P_{\lambda,f})$ with $0 < \lambda < \lambda^*(\Omega, f)$ is radially symmetric. The recursive linear scheme that is used to construct the minimal solutions, gives a radial function at each step, and the resulting limiting function is therefore radially symmetric.

For a solution $u(r)$ on the ball of radius R, we have $u_r(0) = 0$ and

$$-u_{rr} - \frac{N-1}{r}u_r = \lambda f(r)F(u) \quad \text{in} \quad (0, R)\,.$$

Hence, $-\frac{d(r^{N-1}u_r)}{dr} = \lambda f(r)r^{N-1}F(u) \ge 0$, and therefore $u_r < 0$ in $(0, R)$ since $u_r(0) = 0$. This shows that $u(r)$ attains its maximum at 0, and that – just as in the case where $f \equiv 1$ – we have $\|u^*\|_\infty = u^*(0)$. □

It follows from this proposition that for radially symmetric domains Ω and profiles f, the extremal solution u^* is necessarily radially symmetric and that the pull-in distance is nothing but $u^*(0)$. We shall denote by $\lambda^*_\alpha(N)$ (resp., u^*_α) the pull-in voltage (resp., the extremal solution) of $(P_{\lambda,f})$ when $f(x) = |x|^\alpha$, and Ω is the unit ball in $\mathbb{R}^N$.

We now make the following crucial observation.

Proposition 4.2. *For any $\alpha > -2$, the change of variable $u(r) = w(r^{1+\frac{\alpha}{2}})$ gives a correspondence between the radially symmetric solutions of the equation*

$$\begin{cases} -\Delta_N u = \lambda(1+\frac{\alpha}{2})^2|x|^\alpha F(u) & \textit{in } B, \\ \qquad u = 0 & \textit{on } \partial B, \end{cases} \tag{20}$$

in dimension N and those of the equation

$$\begin{cases} -\Delta_{\frac{2(N+\alpha)}{2+\alpha}} w = \lambda F(w) & \textit{in } B, \\ \qquad w = 0 & \textit{on } \partial B, \end{cases} \tag{21}$$

in – the potentially fractional – dimension $N(\alpha) = \frac{2(N+\alpha)}{2+\alpha}$. Moreover, we have

$$\lambda^*_\alpha(N) = (1+\tfrac{\alpha}{2})^2\lambda^*_0(N(\alpha)) \quad \textit{and} \quad u^*_\alpha(r) = w^*(r^{1+\frac{\alpha}{2}}), \tag{22}$$

where u^*_α is the extremal solution for (20) and w^* is the extremal solution of (21).

Proof: Indeed, by noting that for a radially symmetric u, we have $\Delta_N u = u'' + \frac{N-1}{r}u'$, a straightforward calculation gives that

$$\Delta_N u(r) + (1+\frac{\alpha}{2})^2\lambda r^\alpha F(u(r)) = (1+\frac{\alpha}{2})^2 r^\alpha\Big(\Delta_{N(\alpha)}w(r^{1+\frac{\alpha}{2}}) + \lambda F(w(r^{1+\frac{\alpha}{2}})\Big),$$

where $N(\alpha) = \frac{2(N+\alpha)}{2+\alpha}$. The rest follows from the uniqueness of the extremal solutions.

The above transformation allows us to deduce many results for the case of a power-law profile, from corresponding ones associated to constant profiles. The fact that it preserves the L^∞-norm has consequences on the pull-in distance and on the role of the profile in the critical dimension. It does also give proofs for various intriguing phenomena displayed by the numerical results below, especially in the case of a two dimensional disc, where the transformation does not alter the dimension since then $N(\alpha) = 2$.

The following corollary summarizes these consequences.

Corollary 4.1. *With the above notations, the following hold:*

1. *For any dimension $N \geq 1$, we have for $\alpha >> 1$,*

$$\lambda_\alpha^*(N) \sim (1+\frac{\alpha}{2})^2 \lambda_0^*(2). \tag{23}$$

2. *If $N = 2$, then*

$$\lambda_\alpha^*(2) = (1+\tfrac{\alpha}{2})^2 \lambda_0^*(2) \quad \textit{and} \quad \|u_\alpha^*\|_{L^\infty} = \|u_0^*\|_{L^\infty} \textit{ for all } \alpha > -2. \tag{24}$$

Proof. 1) From the above proposition, we have $\lambda_\alpha^*(N) = (1+\frac{\alpha}{2})^2 \lambda_0^*(\frac{2N+2\alpha}{\alpha+2})$, and $\lambda_0^*(\frac{2N+2\alpha}{\alpha+2}) \sim \lambda_0^*(2)$ whenever α is large.
2) follows from the fact that for $N = 2$, we then have $N_\alpha = 2$ for each α which means that

$$\lambda_\alpha^*(2) = \frac{(\alpha+2)^2}{4} \lambda_0^*(2), \tag{25}$$

and the pull-in distance in dimension 2 on the ball is $\|u_\alpha^*\|_{L^\infty} = \|w^*\|_{L^\infty}$, where $u_\alpha^*(r) = w^*(r^{1+\frac{\alpha}{2}})$. The pull-in distance is therefore independent of α. ☐

Corollary 4.2. *The following estimates hold in a MEMS model with a power-law permittivity profile, i.e., if $F(u) = (1-u)^{-2}$ and $f(x) = |x|^\alpha$.*

1. *For any dimension $N \geq 1$, we have for $\alpha >> 1$,*

$$\lambda_\alpha^*(N) \sim 0.789(1+\frac{\alpha}{2})^2. \tag{26}$$

2. *If $N = 2$, then*

$$\lambda_\alpha^*(2) = 0.789(1+\tfrac{\alpha}{2})^2 \quad \textit{and} \quad \|u_\alpha^*\|_{L^\infty} = 0.445 \textit{ for all } \alpha > -2. \tag{27}$$

3. *If $1 \leq N \leq 7$ or if $N \geq 8$ and $\alpha > \alpha_N := \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}}$, then the extremal solution u_α^* of $(M_{\lambda,\alpha})$ on the ball is classical and the pull-in distance $\|u_\alpha^*\|_{L^\infty} < 1$.*

4. *If the dimension $N \geq 8$, and $0 \leq \alpha \leq \alpha_N := \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}}$, then the extremal solution is exactly $u_\alpha^*(x) = 1-|x|^{\frac{2+\alpha}{3}}$, which means that*

$$\lambda_\alpha^*(N) = \tfrac{(2+\alpha)(3N+\alpha-4)}{9} \quad \textit{and} \quad \|u_\alpha^*\|_{L^\infty} = 1. \tag{28}$$

Proof. 1) and 2) follow from the above proposition and the fact that $\lambda_0^*(2) = 0.789$ and $\|u_0^*\|_{L^\infty} = 0.445$.

3) The extremal solution u_α^* of $(M_{\lambda,\alpha})$ is regular if and only if $\|u_\alpha^*\|_{L^\infty} = \|w^*\|_{L^\infty} < 1$, where w^* is the extremal solution for (M_λ) in dimension $N(\alpha)$. According to [10], this happens if $\frac{N(\alpha)}{2} < 1 + \frac{4}{3} + 2\sqrt{\frac{2}{3}}$ which means that $\alpha > \alpha_N := \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}}$.

4) Note first that $u^*(x) = 1-|x|^{\frac{2+\alpha}{3}}$ is a $H_0^1(B)-$weak solution of $(M_{\lambda,|x|^\alpha})$ for any $\alpha > 4-3N$. The voltage is then $\lambda_\alpha(N) = \frac{(2+\alpha)(3N+\alpha-4)}{9}$. Since now $\|u^*\|_{L^\infty} = 1$, then by Proposition 1.1, it remains only to show that for all $\phi \in H_0^1(B)$,

$$\int_B |\nabla\phi|^2 \geq \int_B \frac{2\lambda|x|^\alpha}{(1-u^*)^3}\phi^2. \tag{29}$$

But Hardy's inequality gives for $N \geq 3$ that $\int_B |\nabla\phi|^2 \geq \frac{(N-2)^2}{4}\int_B \frac{\phi^2}{|x|^2}$ for any $\phi \in H_0^1(B)$, which means that (29) holds whenever $2\lambda_\alpha(N) \leq \frac{(N-2)^2}{4}$ or, equivalently, if $N \geq 8$ and $0 \leq \alpha \leq \alpha_N = \frac{3N-14-4\sqrt{6}}{4+2\sqrt{6}}$. ☐

The above scaling has also the following direct consequences.

Corollary 4.3. *Suppose F is a regular nonlinearity and $N < 10 + 4\alpha$, then the extremal functional u^*_α of $(P_{\lambda,\alpha})$ on the ball is classical.*

Proof. Cabre and Cappella [2] showed that the extremal solution on the ball is always bounded for $N \le 9$. They were only interested in integer dimensions, but an inspection of their proof indicates that the same result holds for any fractional dimensions $N < 10$. Combining this with our observation in Proposition 4.2 completes the proof. To see that this is optimal one recalls that when $F(v) = e^v$ the extremal solution is unbounded in $N = 10$. Using this fact and the change of variables above yields the optimality of this result. ☐

Remark 4.1. One can also use this change of variables to study permittivity profiles with negative powers (i.e., $f(x) = |x|^\alpha$ for $0 > \alpha > -2$. For example suppose $F(u) = e^u$, then using the above change of variables, one can show that for a fixed N ($3 \le N \le 9$), the extremal solution associated with

$$-\Delta u = |x|^\alpha e^u \quad \text{ on } B,$$

is singular for $\alpha \in (-2, \frac{10-N}{4}]$, while it is a classical solution for $\alpha \in (\frac{10-N}{4}, 0)$.

5 Asymptotic behavior of stable solutions near the pull-in voltage

We now establish pointwise upper and lower estimates on the minimal solutions u_λ in terms of λ, λ^*, the extremal solution u^* and $\frac{d}{d\lambda} u_\lambda\big|_{\lambda=\lambda^*}$. For simplicity we restrict our attention to $F(u) = e^u$ and $F(u) = (1-u)^{-2}$. In addition we allow fractional dimensions for results on the unit ball since then, one can apply the results of the previous section to deal with power-law profiles $(P_{\lambda,\alpha})$. We first recall that by using Proposition 1.1, one can show the following:

- If $F(u) = e^u$, then $u^*(x) = \log(\frac{1}{|x|^2})$ is an extremal solution on the unit ball in $\mathbb{R}^N$ at $\lambda^* = 2N - 4$, provided $N \ge 10$.
- If $F(u) = \frac{1}{(1-u)^2}$, then $u^*(x) = u^*(x) = 1 - |x|^{\frac{2}{3}}$ is an extremal solution on the unit ball in $\mathbb{R}^N$ at $\lambda^* = \frac{6N-8}{9}$, provided $N \ge \frac{14+\sqrt{6}}{3}$.

Theorem 5.1. *Let u^* denote the extremal solution of (P_λ) on a smooth bounded domain Ω in $\mathbb{R}^N$.*

1. *If $F(u) = (1-u)^{-2}$, then for $0 < \lambda < \lambda^*$, we have*

$$u_\lambda(x) \le \left(\frac{\lambda}{\lambda^*}\right)^{\frac{1}{3}} u^*(x) \qquad \text{for a.e. } x \in \Omega. \tag{30}$$

Moreover, if Ω is the unit ball in $\mathbb{R}^N$ with $N \ge \frac{14+4\sqrt{6}}{3} = 7.93...$, then for $0 < \lambda < \lambda^* = \frac{6N-8}{9}$ we have

$$1 - |x|^{\frac{2}{3}} - \frac{3(\lambda^* - \lambda)}{(6N-8)} \left(|x|^{\frac{-N}{2}+1+\frac{\sqrt{9N^2-84N+100}}{6}} - 1 \right) \le u_\lambda(x) \le \left(\frac{\lambda}{\lambda^*}\right)^{\frac{1}{3}} (1 - |x|^{\frac{2}{3}}), \tag{31}$$

for a.e. $x \in \Omega$.

2. *If $F(u) = e^u$, then for $0 < \lambda < \lambda^*$,*

$$u_\lambda(x) \le \log\left(\frac{\lambda^*}{\lambda^* - \lambda + \lambda e^{-u^*}}\right) \qquad \text{for a.e. } x \in \Omega. \tag{32}$$

Moreover, if Ω is the unit ball in $\mathbb{R}^N$ with $N \geq 10$, then for $0 < \lambda < \lambda^* = 2N-4$ we have

$$\log(\frac{1}{|x|^2}) - \frac{(\lambda^*-\lambda)}{(2N-4)} \left(|x|^{\frac{-N}{2}+1+\frac{\sqrt{N^2-12N+20}}{2}} - 1 \right) \leq u_\lambda(x) \leq \log\left(\frac{\lambda^*}{\lambda^*-\lambda+\lambda|x|^2}\right), \tag{33}$$

for a.e. $x \in \Omega$.

Proof. The upper estimates follow easily from the minimality of u_λ and the fact that $x \mapsto \left(\frac{\lambda}{\lambda^*}\right)^{\frac{1}{3}} u^*(x)$ (resp., $x \mapsto \log\left(\frac{\lambda^*}{\lambda^*-\lambda+\lambda e^{-u^*}}\right)$ is a supersolution of (P_λ) in the case that $F(u) = (1-u)^{-2}$ (resp., $F(u) = e^u$).

For the lower bound, we shall proceed as follows: First, recall that $\lambda \mapsto u_\lambda$ is differentiable and increasing on $(0, \lambda^*)$, and so if one defines $v_\lambda := \frac{d}{d\lambda} u_\lambda$, then v_λ is positive and solves the linear equation

$$\begin{cases} -\Delta v = F(u_\lambda) + \lambda F'(u_\lambda) v & \text{in } \Omega, \\ \quad v = 0 & \text{on } \partial\Omega, \end{cases} \tag{Q_λ}$$

where, F is given by either e^u or $(1-u)^{-2}$. We shall need the following notion.

Definition 5.1. *An extremal solution u^* associated with (P_λ) is said to be* super-stable *provided there exists $\varepsilon > 0$ such that*

$$(\lambda^* + \varepsilon) \int_\Omega F'(u^*)\psi^2 \leq \int_\Omega |\nabla \psi|^2 \qquad \textit{for all } \psi \in H_0^1(\Omega).$$

Note that if u^* is a super-stable extremal solution then $\mu_1(\lambda^*, u^*) > 0$. We shall see at the end of this section that the converse is however not true. We first establish the following result.

Lemma 5.1. *Assume Ω is a smooth bounded domain in $\mathbb{R}^N$. Then,*

1. *For $0 < \lambda < \lambda^*$, v_λ is the unique H_0^1-weak solution of (Q_λ).*
2. *$\lambda \mapsto v_\lambda$ is increasing on $(0, \lambda^*)$, and therefore $v^*(x) := \lim_{\lambda \to \lambda^*} v_\lambda(x)$ is defined for a.e. $x \in \Omega$.*
3. *$\lambda \mapsto u_\lambda$ is convex on $(0, \lambda^*)$, and therefore for $0 < \lambda < \lambda^*$ we have for a.e. $x \in \Omega$,*
$$u_\lambda(x) \geq u^*(x) + (\lambda - \lambda^*) v^*(x). \tag{34}$$
4. *If u^* is super-stable, then v^* is the unique H_0^1-weak solution of $(Q)_{\lambda^*}$.*

Proof. (1) One can use the fact that $\mu_1(\lambda, u_\lambda) \geq 0$, and a standard minimization argument to show the existence of an H_0^1-solution to (Q_λ). Using the fact that $\mu_1(\lambda, u_\lambda) > 0$ one can see that the solution is unique.

(2) Let $0 < \lambda < \lambda^*$ and $\varepsilon > 0$ small. Note first that

$$\begin{aligned} -\Delta(v_{\lambda+\varepsilon} - v_\lambda) &= F(u_{\lambda+\varepsilon}) - F(u_\lambda) + \varepsilon F'(u_{\lambda+\varepsilon}) v_{\lambda+\varepsilon} \\ &\quad + \lambda F'(u_{\lambda+\varepsilon}) v_{\lambda+\varepsilon} - \lambda F'(u_\lambda) v_\lambda \\ &= g(x) + \lambda F'(u_\lambda)(v_{\lambda+\varepsilon} - v_\lambda), \end{aligned}$$

where

$$g(x) := F(u_{\lambda+\varepsilon}) - F(u_\lambda) + \varepsilon F'(u_{\lambda+\varepsilon}) v_{\lambda+\varepsilon} + \lambda\big(F'(u_{\lambda+\varepsilon}) v_{\lambda+\varepsilon} - F'(u_\lambda) v_{\lambda+\varepsilon}\big)$$

is in $H^1(\Omega)$ and is positive. Now set $w := v_{\lambda+\varepsilon} - v_\lambda$ in such a way that w solves

$$\begin{aligned} -\Delta w &= g(x) + \lambda F'(u_\lambda) w \qquad \text{on } \Omega, \\ w &= 0 \qquad\qquad\qquad\qquad \text{on } \partial\Omega. \end{aligned}$$

Testing this equation on w^- gives

$$-\int_\Omega g w^- \geq \mu_1(\lambda, u_\lambda) \int_\Omega (w^-)^2,$$

and hence $w^- = 0$ a.e. in Ω. By the maximum principle one then get that $w > 0$ in Ω and hence that $\lambda \to v_\lambda$ is increasing. We can therefore define the limit $v^*(x) := \lim_{\lambda \to \lambda^*} v_\lambda(x)$, which exists a.e. x in Ω, though it might be infinite on a large set.

(3) The convexity of $\lambda \mapsto u_\lambda$ follows from the fact that $\lambda \mapsto v_\lambda$ is increasing. We can therefore write $u_\lambda \geq u_t + (\lambda - t) v_t$ for $0 < \lambda, t < \lambda^*$ and a.e. $x \in \Omega$. The claim now follows by letting t go to λ^*.

(4) Since u^* is super-stable one has

$$(\lambda + \varepsilon) \int_\Omega F'(u_\lambda)\psi^2 \leq \int_\Omega |\nabla \psi|^2 \qquad \forall \psi \in H_0^1.$$

Using this and testing (Q_λ) on v_λ gives

$$\varepsilon \int_\Omega F'(u_\lambda) v_\lambda^2 \leq \int_\Omega F(u_\lambda) v_\lambda.$$

Since F is either $F(u) = e^u$ or $F(u) = (1-u)^{-2}$, the left hand side is necessarily bounded. From this and again by testing (Q_λ) on v_λ one sees that v_λ is bounded in H_0^1. Passing to limits, one sees that v^* is a H_0^1-weak solution of (Q_{λ^*}). The uniqueness follows from the fact that $\mu_1(\lambda^*, u^*) > 0$. $\square$

We now complete the proof of Theorem 5.1. For that we assume that Ω is the unit ball in $\mathbb{R}^N$. It is then easy to show using Hardy's inequality that the explicit extremal solutions for (P_λ) given above, are super-stable provided $N > 10$ (resp., $N > \frac{14+4\sqrt{6}}{3} = 7.93...$) when $F(u) = e^u$ (resp., $F(u) = (1-u)^{-2}$). An easy calculation also shows that

$$v^*(x) = \frac{1}{2N-4}\left(|x|^{\frac{-N}{2} + 1 + \frac{\sqrt{N^2 - 12N + 20}}{2}} - 1 \right),$$

(when $F(u) = e^u$) resp.,

$$v^*(x) = \frac{3}{6N-8}\left(|x|^{\frac{-N}{2} + 1 + \frac{\sqrt{9N^2 - 84N + 100}}{6}} - 1 \right),$$

(when $F(u) = (1-u)^{-2}$) are H_0^1-weak solutions of $(Q)_{\lambda^*}$ in the respective cases, assuming the dimension restrictions above. Using this and the earlier convexity result gives the desired lower bounds for $N > 10$ ($N > \frac{14+4\sqrt{6}}{3}$) in the exponential and MEMS cases respectively. To obtain the result for the critical dimensions one passes to the limit in N. We omit the details. $\square$

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